Announcements

1) Quiz next week,
Exam 2 week after
(both on Thursdays)

Second Order Linear Equations with Nonconstant Coefficients

(Section 4.7)

In principle, we know how to solve

$$ay'' + by' + cy = f$$

for a, b, c constants.

Example 1: Solve

$$t^2 y'' + y = \cos(t).$$

Can't reduce to an equation with constant coefficients.

Again observe that if f_1 and f_2 are two solutions,

$$t^2 f_1'' + f_1 = \cos(t)$$
$$t^2 f_2'' + f_2 = \cos(t)$$

$$t^2 f_1'' + f_1 = \cos(t)$$
$$t^2 f_2'' + f_2 = \cos(t),$$

we can subtract to get

$$t^2(f_1'' - f_2'') + (f_1 - f_2) = 0$$

which is

$$t^2(f_1 - f_2)'' + (f_1 - f_2) = 0.$$

So $g = f_1 - f_2$ is a solution to the homogeneous system

$$\boxed{t^2 y'' + y = 0}$$

This means we should first try to solve the homogeneous system, then find one solution to the original. How do we solve the homogeneous system?

$$t^2 y'' + y = 0$$

Wish Upon A Star

Suppose a solution to

$t^2 y'' + y = 0$ is given

by $\boxed{y = t^r}$ for some

power r.

Then $y' = r t^{r-1}$ $(r \neq 0)$

$$y'' = r(r-1) t^{r-2}$$

Then substituting,

$$t^2\left(r(r-1)t^{r-2}\right) + t^r = 0$$

$$(r^2-r)t^r + t^r = 0$$

Factor out t^r

$$t^r\left(r^2-r+1\right) = 0$$

Solve $r^2-r+1=0$

(we need to do this since $t^r=0$ only for $r=0$)

For $r^2 - r + 1 = 0$, the solutions are

$$r = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm i\sqrt{3}}{2}$$

Solutions of the form

$$\boxed{t^{\frac{1 \pm i\sqrt{3}}{2}}}$$

$$\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$= \cos\left(\frac{\pi}{3}\right) \pm i\sin\left(\frac{\pi}{3}\right)$$

$$= e^{\pm i\frac{\pi}{3}}$$

How to decompose

$$t^{\frac{1}{2} \pm i\frac{\sqrt{3}}{2}} \; ?$$

$$t^{\frac{1}{2} \pm \frac{i\sqrt{3}}{2}} = t^{1/2}\left(t^{\pm i\frac{\sqrt{3}}{2}}\right)$$

real

How to decompose

$$t^{\pm \frac{i\sqrt{3}}{2}} \;?$$

Can write

$$t^{\pm \frac{i\sqrt{3}}{2}} = e^{\ln\left(t^{\pm \frac{i\sqrt{3}}{2}}\right)}$$

$$= e^{\pm \frac{i\sqrt{3}}{2}\ln(t)}$$

(works, but requires loads of justification!)

$$e^{\pm i \frac{\sqrt{3}}{2} \ln(t)}$$

$$= \cos\left(\frac{\sqrt{3}}{2}\ln(t)\right) \pm i \sin\left(\frac{\sqrt{3}}{2}\ln(t)\right)$$

Same Result From Constant Coefficients Applies:

The real and imaginary parts are solutions to

$$t^2 y'' + y = 0.$$

So solutions to

$t^2 y'' + y = 0$ are of the

form

$$C_1 t^{1/2} \cos\left(\frac{\sqrt{3}}{2} \ln(t)\right)$$

$$+ C_2 t^{1/2} \sin\left(\frac{\sqrt{3}}{2} \ln(t)\right)$$

Check!

Just do $t^{1/2} \sin\left(\frac{\sqrt{3}}{2} \ln(t)\right)$

Use a computer!

Q: How to find solutions of the nonhomogeneous system?

A: Variation of Parameters!

Variation of Parameters Again

Suppose that there is a single solution $y_p(t)$ to $t^2 y'' + y = \cos(t)$.

Wish that

$$\boxed{y_p(t) = u(t)y_1(t) + v(t)y_2(t)}$$

with $y_1(t) = t^{1/2}\cos\left(\frac{\sqrt{3}}{2}\ln(t)\right)$

$y_2(t) = t^{1/2}\sin\left(\frac{\sqrt{3}}{2}\ln(t)\right)$

Wish further that

$$u'y_1 + v'y_2 = 0.$$

Plug into $t^2 y'' + y = \cos(t)$.

$$y' = u y_1' + v y_2'$$

$$y'' = u' y_1' + u y_1'' + v' y_2' + v y_2''$$

We get

$$\overbrace{u t^2 y_1'' + u y_1}^{=0} + \overbrace{v t^2 y_2'' + v y_2}^{0=}$$
$$+ \; t^2(u' y_1' + v' y_2') = \cos(t)$$

Now have equations

$$u'y_1 + v'y_2 = 0$$

$$u'y_1' + v'y_2' = \frac{\cos(t)}{t}$$

Solve for u, v!